Section 2.1. Solution Curve Without a Solution

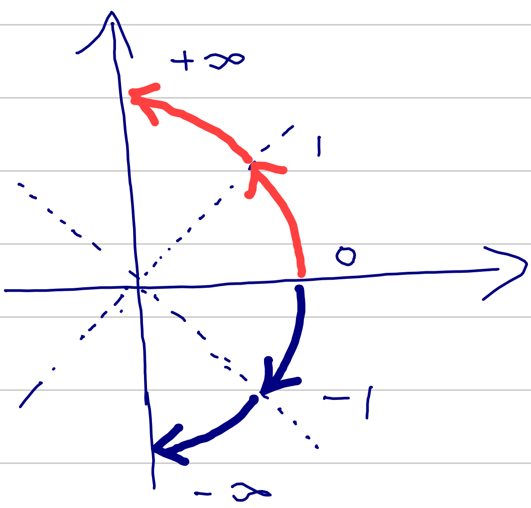
1. DIRECTED FIELDS.

As we have seen in Section 1.2 that whenever $f(x,y)$ and $\partial f/\partial y$ satisfy certain continuity conditions, qualitative question about existence and uniqueness of solution can be answered.

In this section we will see that other qualitative question about properties of solutions:

- How does a solution behave near a certain point?
- How does a solution behave asympotically?

Recall: A derivative $\frac{dy}{dx}$ of a differentiable function $y = y(x)$ gives slopes of tangent lines at points on it graph.



Consider a first-order ODE (in the normal form)

$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

The function $f(x,y)$ is usually called the **slope function** or the **rate function.**

If we evaluate f over a rectengular grid of points in the xy-plane, and draw a small arrow, called a **lineal element,** at each point (x,y) of the grid with slope $f(x,y)$. (A line element is usually oriented by increasing direction of x.) Then the collection of all these line elements is called a **direction field** (or a **slope field**) of the ODE $dy/dx = f(x,y)$

Visually, the direction field suggests the shape of a family of solution curves of the diff. eq., and consequently, it may be possible to see at glance certain qualitative aspects of the solutions - regions in the plane, e.g., in which a solution has unusual behavior.

A single solution curve passes through a direction field must follow the flow pattern of the field: it tangent to a lineal element when it intersects a point of the grid.

In general, a finner grid gives a better approx of the solution curve.

Example: The direction field of the equation

$$\frac{dy}{dx} = 0.2xy$$

can tell us when $|f(x,y)|$ increases as $|x|$ and y increase, when $f(x,y)$ is positive / negative.

This suggest the behavior of a solution curve.

Example: The direction field of the diff. eq.

$$\frac{dy}{dx} = \sin(x+y).$$

Example The direction field of

$$\frac{dy}{dx} = \sin y.$$

2. Autonomous First-Order DEs.

An ODE in which the independent variable does not appear explicitly is said to be **autonomous**. If x denotes the independent variable, then an autonomous ODE has form

$$F(y, y', y'', \dots) = 0.$$

In particular, a first-order autonomous ODE has form

$$F(y, y') = 0$$

or

$$\frac{dy}{dx} = f(y). \qquad (2)$$

For example, $\frac{dy}{dx} = \sin y$ is autonomous while $\frac{dy}{dx} = 4xy^{1/2}$ is nonautonomous.

All the equations in section 1.3 are autonomous

$$\frac{dP}{dt} = kP, \quad \frac{dx}{dt} = kx(n+1-x), \quad \frac{dT}{dt} = k(T - T_m)$$

$$\frac{dA}{dt} = 12 - \frac{A}{50}, \ldots$$

CRITICAL POINTS. The zeros of a first-order autonomous DE (2) are very important.

We say that a real number c is a **critical point** of the eq (2) if it is a zero of f. ($f(c)=0$).

A critical point is also called an **equilibrium point** or a **stationary point**. Observe that when we plug in $y(x)=c$ into (2) then both sides of the equation are zero. This means:

"If c is a critical point of (2), then $y(x)=c$ is a constant solution of the autonomous DE."

The solution $y(x)=c$ is called an **equilibrium solution**.

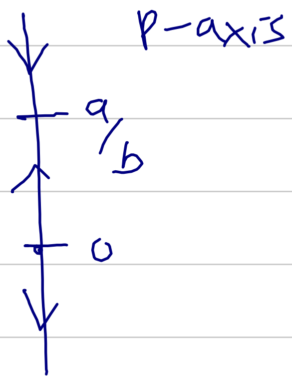
Example: Consider the diff. eq.

$$\frac{dP}{dt} = P(a - bP),$$

where a, b are positive constant.

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	$-$	$\searrow$	$\downarrow$
$(0, a/b)$	$+$	$\nearrow$	$\uparrow$
$(a/b, \infty)$	$-$	$\searrow$	$\downarrow$

$\Rightarrow$ Phase portrait of DE:

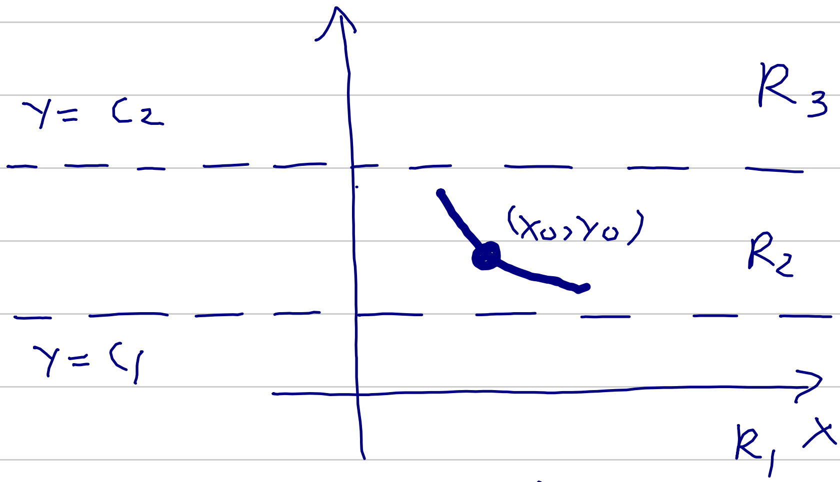


SOLUTION CURVE.

Without solving an autonomous DE, we can say pretty much about its solution curves.

- Since the function f in (2) is independent from x, we can consider f defined for $-\infty < x < \infty$ or for $0 \leq x < \infty$.
- Since f and f' are continuous functions of y on some interval I of the y-axis, Theorem 1.2.1 hold for some horizontal strip R corresponding to I, so through any point $(x_0, y_0) \in R$ there passes only one solution curve of (2).
- for sake of discussion, we assume that (2) has exactly two equilibrium $y(x) = c_1$ and $y(x) = c_2$ for $c_1 < c_2$. These solution curves partition

the plane into 3 parts R_1, R_2, R_3.



Without proof we can say the follows about nonconst solution $y(x)$ of (2):

+ If $(x_0, y_0) \in R_i$ $(i = 1, 2, 3)$, and $y(x)$ is a solution whose graph passes through (x_0, y_0), then the graph remain in R_i for all x.

+ By the continuity of f, we must have either $f(y) > 0$ or $f(y) < 0$ for all y in R_i.

+ Since $dy/dx = f(y(x))$ is either positive or negative in R_i, $y(x)$ is strictly monotonic.

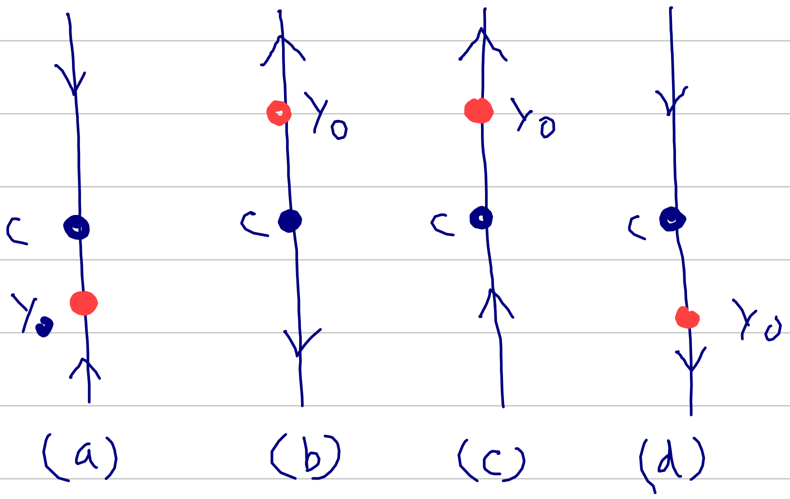
+ If $y(x)$ is bounded above by a critical point c_1, then the graph of $y(x)$ must approach the graph of $y(x) = c_1$ either as $x \to \infty$ or as $x \to -\infty$

<u>Example</u>: Solution of $\dfrac{dy}{dx} = (y-1)^2$.

ATTRACTORS AND REPELLERS.

Suppose that $y(x)$ is a nonconstant solution of the autonomous differential equation given in (2) and that c is a critical point of the DE.

There are 3 types of behavior that $y(x)$ can be exhibit near c.



(a) Two arrows point toward c.

→ All solution $y(x)$ of (2) that start from an initial point (x_0, y_0) sufficiently near c exhibit the asymptotic behavior $\lim_{x\to\infty} y(x) = c$. We say c is **asymptotically stable**, and c is also called an **attractor**.

(b) Two arrows point away c.

⇒ All solutions starting from (x_0, y_0) will move away from c as x increases.

So, x is **unstable**, and x is called a **repeller**.

(c) and (d) c is **semi-stable**.

AUTONOMOUS DEs AND DIRECTION FIELD

- All lineal elements along a horizontal strip have the same slope.
- The lineal elements along a vertical strip is vary.

TRANSLATION PROPERTY.

If $y(x)$ is a solution of $dy/dx = f(y)$, then $y_1(x) = y(x-k)$, k is a constant, is also a solution.

Example: $y(x) = e^x$ is a solution of $y' = y$.

Of course $y(x) = e^{x+6}$ is also a solution.